QFI Quantitative Finance Exam

# Exam QFIQF

**Date:** Monday, October 26, 2020

## INSTRUCTIONS TO CANDIDATES

**General Instructions**

1. This examination has 15 questions numbered 1 through 15 with a total of 100 points.

   The points for each question are indicated at the beginning of the question.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the supervisor or proctor cannot give you any guidance beyond the instructions provided in this document.

3. Each question part or subpart should be answered either in the Word document or the paper provided as directed. Graders will only look at the work as indicated.

4. In the Word document, answers should be entered in the box marked ANSWER. The box will expand as lines of text are added. There is no need to use special characters or subscripts (though they may be used). For example, $\beta_1$ can be typed as beta_1 (and ^ used to indicate a superscript).

   The Word file that contain your answers must be uploaded before time expires.

**Written-Answer Instructions**

1. Write your candidate number at the top of each sheet. Your name must not appear.

2. Write on only one side of a sheet. Start each question on a fresh sheet. On each sheet, write the number of the question that you are answering. Do not answer more than one question on a single sheet.

3. The answer should be confined to the question as set.

4. When you are asked to calculate, show all your work including any applicable formulas. When you are asked to recommend, provide proper justification supporting your recommendation.

5. When you finish, hand in all your written-answer sheets to the Prometric center staff. Be sure to hand in all your answer sheets because they cannot be accepted later.

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*The responses for all parts of this question are required on the paper provided to you.*

**1.** (*6 points*) Let $\{W_t : t \geq 0\}$ be a standard Wiener process and $M_t = \int_0^t f(s)dW_s$, where $f(s)$ is a square integrable deterministic function.

Let $X_t = e^{-\theta M_t - \frac{1}{2}\theta^2 \int_0^t f(s)^2 ds}$, where $\theta$ is a constant.

(a) (*1.5 points*) Show that $X_t$ satisfies the stochastic differential equation

$$dX_t = \theta f(t) X_t dW_t$$

(b) (*1 point*) Show that $M_t \sim \text{Normal}(0, \int_0^t f(s)^2 ds)$ for any $t > 0$.

Suppose that $Z_t$ satisfies the stochastic differential equation

$$dZ_t = \frac{y - Z_t}{1-t} dt + dW_t \text{ for } 0 \leq t < 1,$$

with $Z_0 = z$ and $Z_1 = y$ where both $y$ and $z$ are constants.

(c) (*2 points*) Show that $Z_t = yt + (1-t)(z + \int_0^t \frac{1}{1-s} dW_s)$ for $0 \leq t < 1$.

(d) (*1 point*) Find the mean and the variance of $Z_t$ for $0 \leq t < 1$.

(e) (*0.5 points*) Show that $Z_t$ follows a normal distribution for $0 < t < 1$.

*The responses for all parts of this question are required on the paper provided to you.*

**2.** (*6 points*) You are an actuary working in the asset liability management department. Recently, you were assigned to examine nontraditional annuities, including equity indexed annuities.

(a) (*2 points*) Describe four critical provisions that are common to equity indexed annuities contracts.

The guarantees embedded in equity indexed annuities are similar to basket options. Your manager asks you to look at basket options. In particular, you are asked to look at a European call option on an index that consists of two stocks.

Denote the index price by

$$S_t = X_t + Y_t,\ 0 \le t \le T,$$

where $T$ is the maturity, $X_t$ and $Y_t$ are the prices of two stocks.

Let $K$ be the strike of the basket option. Under the risk-neutral measure $\mathbb{Q}$, the European basket call price $C$ can be calculated by $C = e^{-rT} E^{\mathbb{Q}}\left[ f(S_T - K) \right]$, where $r$ denotes the constant risk-free interest rate and $f(x) = \max(x, 0)$.

(b) (*1.5 points*) Show that $C \ge f(S_0 - e^{-rT} K)$ using Jensen's inequality.

**2. Continued**

Suppose that $X_t$ and $Y_t$ satisfy the following stochastic processes under the real-world measure $\mathbb{P}$ :

$$dX_t = \mu_1 X_t dt + \sigma_1 X_t dW_t , \quad X_0 = 1$$
$$dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dW_t , \quad Y_0 = 1$$

where $\mu_1$, $\mu_2$, $\sigma_1 > 0$, $\sigma_2 > 0$ are constants and $W_t$ is a standard Wiener process under $\mathbb{P}$ .

(c) (*2 points*) Establish a condition on $\mu_1$, $\mu_2$, $\sigma_1$ and $\sigma_2$ such that both $X_t e^{-rt}$ and $Y_t e^{-rt}$ are martingales under the risk-neutral measure $\mathbb{Q}$ .

(d) (*0.5 points*) Derive the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ by assuming that the condition in part (c) holds.

*The responses for all parts of this question are required on the paper provided to you.*

**3.** (*7 points*) Let $\{W_t : t \geq 0\}$ be a standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be the partitioning of the interval $[0, T]$ into $n$ smaller subintervals all of size $h$ for an arbitrary integer number $n > 0$ and an arbitrary real number $T > 0$.

Denote by

$$\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}, \qquad i = 0,1, \dots, n-1$$

(a) (*2 points*) Show that for all $i, j = 0, 1, \dots, n-1$

(i) $E\left[\left(\Delta W_{t_i}\right)^4\right] = 3h^2$ using Ito's lemma.

(ii) $E\left[\left(\Delta W_{t_i}\right)^2 \left(\Delta W_{t_j}\right)^2\right] = h^2$ if $i < j$.

(b) (*2 points*) Show that

$$\frac{W_T^2}{2} - \frac{T}{2} - \sum_{i=0}^{n-1} W_{t_i} \Delta W_{t_i} = \frac{1}{2}\left(-T + \sum_{i=0}^{n-1} \left(\Delta W_{t_i}\right)^2\right)$$

(c) (*3 points*) Show that $\frac{W_T^2}{2} - \frac{T}{2}$ is the mean square limit of the following series using parts (a) and (b) above

$$V_n = \sum_{i=0}^{n-1} W_{t_i} \Delta W_{t_i}$$

*The responses for all parts of this question are required on the paper provided to you.*

**4.** (*6 points*) Your colleague has recently used the LIBOR Market Model (LMM) to price a cap and is interested in applying the same framework for pricing a swaption, arguing that for each interest rate derivative, the price has a closed-form solution equal to the Black formula.

Under the LMM framework, you are given the stochastic dynamics of the forward rate $F_i(t)$ under the $\mathbb{Q}^{T_i}$ - measure

$$dF_i(t) = \sigma_i F_i(t) dW_i(t), \qquad i = 0, 1, \ldots, n$$

where

- $W_t(t)$ is a $\mathbb{Q}^{T_i}$ - Brownian motion and $I_t$ - measurable
- $\sigma_i$ is a positive constant
- $T_0 > t$, and $T_i - T_{i-1} = \tau$ for all $i$
  where $\tau$ is a positive constant

(a) (*0.5 points*) Explain two reasons why a market practitioner would prefer LMM to the Heath-Jarrow-Morton framework for pricing interest rate derivatives.

(b) (*0.5 points*) Critique your colleague's intention to use lognormal distributions for the forward rates and forward swap rates when pricing caps and swaptions.

(c) (*0.5 points*) State the three conditions for a stochastic process to be a martingale

(d) (*1 point*) Define $F_{i+1}(t)$ in terms of the zero-coupon bonds $P(t,T_i)$ and $P(t,T_{i+1})$ and explain for which measure/numeraire-pair $F_{i+1}(t)$ is a martingale.

## 4. Continued

For an $I_t$ - adapted process $X_u$, define $Z_t = \exp\left\{-\int_0^t X_u dW_i(u) - \frac{1}{2}\int_0^t X_u^2 du\right\}$. Assume that $Z_t$ is integrable under $\mathbb{Q}^{T_i}$.

(e) (*2 points*) Show that $Z_t$ satisfies the three conditions in part (c) under $\mathbb{Q}^{T_i}$ using Ito's Lemma.

You are given another stochastic process $G_t = \frac{P(t,T_i)}{P(0,T_i)} \times \frac{P(0,T_{i+1})}{P(t,T_{i+1})}$.

(f) (*1.5 points*) Show that $dG_t = \frac{\tau\sigma_{i+1}F_{i+1}(t)}{1+\tau F_i(t)} G_t dW_{i+1}(t)$.

**5.** (*6 points*) You are a pricing actuary at ABC Life helping to design a new Variable Annuity (VA) with target and managed volatility funds.

(a) (*0.5 point*) Explain the impact on the dynamic rebalancing strategy if the equity volatility estimator overestimates the volatility.

| ANSWER: |
|---|

Suppose that you are to assess the target volatility mechanism using a Stochastic Volatility Jump Diffusion (SVJD) model.

(b) (*1 point*) Under an SVJD model,

(i) Explain the effectiveness of the dynamic rebalancing strategy.

| ANSWER: |
|---|

(ii) Explain the impact of volatility jumps on guarantee cost.

| ANSWER: |
|---|

## 5. Continued

A consultant who assists you on designing the new product provided several insurer metrics (table below) for various volatility management strategies based on model simulations. The model simulations used an SVJD model. Your manager is interested in understanding the impact of using the Heston model on these metrics.

| | Capped Volatility | Target Volatility | Capital Preservation | VIX-indexed fees |
|---|---|---|---|---|
| Reduction in volatility cost versus static 60/40 strategy | 15% | 61% | 94% | 26% |
| Vega | 0.40% | 0.12% | 0.03% | 0.36% |
| Equity Volatility | 11.05% | 8.19% | 5.26% | 12.92% |
| Equity Allocation | 59% | 55% | 33% | 60% |
| Prospective Fees Paid | 100 | 100 | 100 | 109 |

(c) (*2 points*) Assess how the model results could change if the Heston model were used instead.

ANSWER:

Currently, your company's hedging model does not include dynamic lapses.

(d) (*1 point*) Explain how dynamic lapses impact VA fees collected and guarantee cost as the moneyness of the guarantee changes.

ANSWER:

(e) (*1 point*) Assess whether it is advantageous for your company to hedge dynamic lapsation risk if you are unsure about the exact moneyness level at which the policyholder exercises the option to surrender.

ANSWER:

**5. Continued**

(f) (*0.5 points*) Suggest a management action to lower the lapsation risk.

| ANSWER: |
| --- |
| |

*The responses for all parts of this question are required on the paper provided to you.*

**6.** (*8 points*) Consider an exotic option $A$ with the following payoff:

- $K+S$, if $S>2K$
- $2S-K$, if $K<S\leq 2K$
- $2(K-S)+S$, if $S\leq K$

where $S$ is the underlying asset paying no dividend and $K$ is a constant.

Let $B$ denote the portfolio consisting of one long position in option $A$ and one short position in the underlying asset $S$, i.e. $B=A-S$.

(a) (*1.5 points*)

(i) Sketch the payoff graph for the portfolio $B$.

(ii) Construct a static hedging strategy for option $A$, with plain vanilla options and the underlying asset *S*.

(b) (*1 point*) Construct a dynamic delta-hedging strategy for this exotic option $A$.

(c) (*1 point*) List pros and cons of static hedging strategies and dynamic hedging strategies.

Assume that $S$ follows

$$dS=\mu_s S dt+\sigma_s S dZ$$

where $\mu_s$ and $\sigma_s$ are deterministic, and $Z$ is a standard Wiener process.

Given a self-financing portfolio $\Sigma = a\left[V-\frac{\partial V}{\partial S}S\right]$, where

- $V$: general option on the underlying asset $S$
- $a$: rebalancing factor, satisfying $\left[V-\frac{\partial V}{\partial S}S\right]da = aSd\left[\frac{\partial V}{\partial S}\right]$

(d) (*1.5 points*) Show that $rV=\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_S^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}$ using the law of one price and Ito's Lemma, where $r$ denotes the constant risk-free rate.

## 6. **Continued**

Assume that Black-Scholes-Merton (BSM) holds, and the option $V$ is delta-hedged.

Use the following notations:

- $\sigma_{s,I}$ : implied volatility
- $\sigma_{s,R}$ : realized volatility
- $\Delta_I$ : delta based on implied volatility
- $\Delta_R$ : delta based on realized volatility
- $\Gamma_I$ : gamma based on implied volatility

(e) (*3 points*)

(i) Show that the profit and loss function P&L of the hedged portfolio satisfies the following when the hedge is constructed with realized volatility $\sigma_{s,R}$ :

$$d(P\&L) = \frac{1}{2}S^2\Gamma_I\left[\sigma_{s,R}{}^2 - \sigma_{s,I}{}^2\right]dt + (\Delta_I - \Delta_R)\left[(\mu_s - r)Sdt + \sigma_{s,R}SdZ\right]$$

(ii) Determine $d(P\&L)$, if $V$ is hedged with implied volatility $\sigma_{s,I}$ instead.

(iii) Describe key P&L characteristics, when hedging with realized volatility vs. implied volatility, using the results in parts (i) and (ii) to support your answer.

**7.** (*8 points*) You are a pricing actuary of an insurance company.

Your company is considering offering an index annuity product where interest rate credited is max (0%, min (the index return of T&T 400, a cap)).

Currently you are considering offering an annual cap of 5%. The index annuity is renewable annually.

Your company is planning to replicate the index credit using options.

You asked a junior analyst on your team to collect the information to price the replicating portfolio. He has provided the following:

- Risk free interest rate is 3.0% per annum.
- Dividend rate is 0.0% per annum.
- The current T&T 400 level is 100.
- Based on the implied volatility (IV) for options of 1year maturity, he has fitted an implied volatility function IV(K) = 15% + (K-100) *1.4%, where K is the strike level.
- Your company is budgeting 0.5% of the initial contract deposit for the replicating portfolio.

(a) (*2 points*)

(i) Explain volatility smiles.

| ANSWER: |
|---|

(ii) Compare the common two approaches that describes the volatility smiles.

| ANSWER: |
|---|

(iii) Explain which approach was provided by the junior analyst.

| ANSWER: |
|---|

**7. Continued**

(b) (*1 point*) Describe the most salient characteristics of the equity volatility smile.

| ANSWER: |
|---|

(c) (*0.5 points*) Identify the trades of the replicating portfolio.

| ANSWER: |
|---|

For part (d), identify the formula used (with reference to the formula # in the Formula Sheet as provided.)

(d) (*2 points*) Calculate the price for the replicating portfolio and determine whether the budget is sufficient for the hedging, using the fitted implied volatility function IV(K) provided.

| *The response for this part is required on the paper provided to you.* |
|---|

(e) (*1 point*) Explain the reasonableness of the implied volatility function IV(K) in the context of smile arbitrage.

| ANSWER: |
|---|

As an improvement to the current product offering, the product actuary would like to add a guarantee so that the index credit cap would be at least 3%. The cap is reset so that the total replication cost equals the 0.5% budget.

(f) (*1.5 points*)

(i) Identify types of market conditions that would negatively affect the ability to manage the product with the added guarantee.

| ANSWER: |
|---|

(ii) Suggest a modeling approach to better measure the risk.

| ANSWER: |
|---|

*The responses for all parts of this question are required on the paper provided to you.*

**8.** (*9 points*) Consider the Cox, Ingersoll and Ross (CIR) model for the short-term interest rate $r_t$:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sqrt{ar_t}\,dX_t \text{ for } t > 0 \text{ with } r_0 > 0$$

where $\gamma, \bar{r}$, and $a$ are constants such that $\gamma\bar{r} > \frac{1}{2}a$, $a > 0$, and $X_t$ is a standard Brownian motion.

(a) (*1 point*) Explain why interest rates are always positive in this model.

(b) (*1.5 points*) Show that $r_t = e^{-\gamma t}r_0 + \bar{r}(1 - e^{-\gamma t}) + \sqrt{a}e^{-\gamma t}\int_0^t e^{\gamma s}\sqrt{r_s}\,dX_s$ .

(c) (*2 points*) Determine $E\left[r_t\right]$ and $Var\left[r_t\right]$.

(d) (*1 point*) Explain whether the CIR model belongs to the class of generalized affine models.

Recall that the price at time $t$ of a zero-coupon bond with \$1 principal with maturity date $T$ is given by

$$Z(r_t, t, T) = e^{A(t,T) - B(t,T)r_t} ,$$

where $A(t,T)$ and $B(t,T)$ are functions of $t$ and $T$ and $A(T,T) = B(T,T) = 0$.

(e) (*1 point*) Express $\frac{\partial Z}{\partial t}$, $\frac{\partial Z}{\partial r}$ and $\frac{\partial^2 Z}{\partial r^2}$ in terms of $Z(r_t, t, T)$, $A(t,T)$ and $B(t,T)$.

(f) (*2.5 points*) Show that

(i) $$\frac{\partial A}{\partial t} = \gamma\bar{r}B(t,T)$$

(ii) $$\frac{\partial B}{\partial t} = \gamma B(t,T) + \frac{1}{2}aB(t,T)^2 - 1$$

**9.** (*4 points*) The financial institution that you work for has issued a fixed annuity that receives $1 million premium at initiation and pays $20,000 every six months for 30 years. To meet the annuity contract's future cash flow obligations, you want to implement a strategy to immunize the interest rate risk.

You are given the following information:

- The overnight deposit rate is 1.24% (annualized simple rate) but could rise or fall in the future.
- The 30-year per-value bond you want to use has a fixed 4% coupon rate, paid semiannually, and has a duration of 19.93.
- The duration of the fixed annuity is 13.91.

(a) (*1 point*) Describe an immunization strategy that you can implement to meet the annuity commitment.

| ANSWER: |
|---|

(b) (*1 point*) Determine the cash flows after six months, assuming that the overnight deposit rate stays the same during the period.

| ANSWER: |
|---|

(c) (*1 point*) Explain why the immunization strategy would work.

| ANSWER: |
|---|

(d) (*1 point*) Explain whether Cash Flow Matching is a viable strategy. Justify your conclusion by identifying its drawbacks or benefits in this context.

| ANSWER: |
|---|

**10.** (*6 points*) You were asked to construct a self-financing delta-rho dynamic hedging strategy for a portfolio of variable annuities (VA) contracts. Assume that:

- $L_t$ is the net value of the VA contracts at quarter $t$
- $\Pi_t$ is the value of the hedging portfolio at quarter $t$
- At time 0, $L_0 = 0$
- The bank account's interest rate for one quarter (or 3 months) is 0.5%

You borrowed $500M to establish the strategy with the following asset choices. At quarter 1, the value of the hedging portfolio has increased to $4.5M.

| Time ($t$) in quarter | Stock | Zero-coupon Bond |
|---|---|---|
| 0 | $200 | $100 |
| 1 | $203 | $101 |

For part (a), identify the formula used (with reference to the formula # in the Formula Sheet as provided.)

(a) (*1 point*) Calculate the position in each of the three assets at time 0.

*The response for this part is required on the paper provided to you.*

(b) (*0.5 points*) Define the objective of the hedging strategy in terms of the insurer's hedged loss at maturity.

ANSWER:

Suppose that the hedge position is determined based on the Black-Scholes-Vasicek (BSV) model for parts (c) and (d).

(c) (*0.5 points*) State one problem with using the forward-looking approach to calibrate the stock volatility.

ANSWER:

**10. Continued**

Next, 100,000 daily market scenarios were projected under each of the three different market models below (where the Brownian motions driving the interest rate and equity processes are independent of each other). These models are data-generating models used to simulate the impact of model risk on the insurer's hedged losses (where model risk refers to deviation between the data-generating model and insurer's hedging model).

| Model | Equity Model | Interest Rate Model |
|---|---|---|
| A | Black-Scholes | One-factor CIR |
| B | Black-Scholes | Three-factor CIR |
| C | Heston | Three-factor CIR |

The table below presents the Conditional Tail Expectation (CTE) 95% measure of the insurer's hedged loss at maturity under each market model. However, the names of the corresponding market models are not disclosed in the results below.

| **Table** | Model X | Model Y | Model Z |
|---|---|---|---|
| CTE 95% | 1.8 | 0.5 | 4.0 |

(d) (*2 points*)

(i) Identify the sources of model risk in your hedging strategy under each of Models A, B, and C.

ANSWER:

(ii) Identify the corresponding market model by matching Model X, Y, and Z to Model A, B, or C. Justify your answer.

ANSWER:

**10. Continued**

You asked your actuarial student to graph the insurer's hedged loss against the stock market volatility for each market model (i.e., Model A, B, C). In the report, you noticed the insurer's hedged loss is mostly centered at zero (without any trends) in all three models.

(e) (*1 point*) Explain whether you agree with the student's result.

| ANSWER: |
| --- |

You expect the interest rate will steadily rise throughout the term of the VA contracts.

(f) (*1 point*)

(i) Explain how a delta-only hedging strategy would affect the insurer's hedged loss if your expectation becomes a reality.

| ANSWER: |
| --- |

(ii) Explain how a wrong expectation would affect the insurer's hedged loss after modifying the hedge strategy.

| ANSWER: |
| --- |

*The responses for all parts of this question are required on the paper provided to you.*

**11.** (*8 points*) Assume that the underlying asset $S_t$ follows the Heston model:

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \sigma_t dX_t \\ d(\sigma_t^2) = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dY_t \end{cases}$$

where *r* is the risk-free interest rate, *k*, $\theta^2$ and $\gamma$ are constants, and $X_t$ and $Y_t$ are standard Wiener processes under the risk-neutral measure $\mathbb{Q}$ with constant correlation $\rho$.

It is known that if *r*=0, the profit and loss on a hedged call with strike $K$ and maturity $T$ can be approximated by $dC = \frac{1}{2}\frac{\partial^2 C_{BSM}}{\partial \sigma^2} E\left[(d\sigma_t)^2\right] + \frac{\partial^2 C_{BSM}}{\partial S_t \partial \sigma} E\left[dS_t d\sigma_t\right]$, where $\frac{\partial^2 C_{BSM}}{\partial \sigma^2}$ and $\frac{\partial^2 C_{BSM}}{\partial S_t \partial \sigma}$ are the call's Volga and Vanna, respectively, under the standard Black-Scholes-Merton model.

(a) (*1.5 points*) Explain why you would expect $\rho$ to be positive or negative.

Let $\sigma_R^2$ be the realized variance measured continuously over the time interval $[0,T]$, that is,

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

For a variance swap with no exchange of cash at time 0, the strike $\sigma_R^2$ is given by $\sigma_K^2 = E_0^{\mathbb{Q}}\left[\sigma_R^2\right]$, where $E_t^{\mathbb{Q}}$ denotes the expectation at time *t* under the measure $\mathbb{Q}$.

(b) (*2 points*)

(i) Show that $d\left(E_0^{\mathbb{Q}}\left[\sigma_t^2\right]\right) = k\left(\theta^2 - E_0^{\mathbb{Q}}\left[\sigma_t^2\right]\right)dt$.

(ii) Show that $E_0^{\mathbb{Q}}\left[\sigma_t^2\right] = e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2$.

(iii) Calculate $\sigma_K^2$ in terms of $k$, $\theta$, $\sigma_0$, and $T$.

## 11. Continued

For each strike break point $S^*$ we construct a portfolio with an infinite number of options, consisting of puts with strikes between 0 and $S^*$ and calls with strikes greater than $S^*$. The value of this portfolio is given by

$$\pi(S,S^*) = \int_0^{S^*} \frac{1}{K^2} P(K,T)dK + \int_{S^*}^{\infty} \frac{1}{K^2} C(K,T)dK$$

where $C(K,T)$ and $P(K,T)$ represent call and put options, respectively, on stock $S_t$ with strike $K$ and maturity $T$:

(c) (*3 points*)

(i) Show that $\sigma_R^2 = \frac{2}{T}\left[\int_0^T \frac{1}{S_t} dS_t - ln\frac{S_T}{S_0}\right]$.

(ii) Describe a continuous replication strategy using $S_t$ and a portfolio $\pi\left(S,S^*\right)$ with the appropriate choice of $S^*$ to capture the realized variance $\sigma_R^2$.

Assume that the true dynamics of variance is stochastic and represented by the Heston model. For a variable annuity contract with guaranteed cash flows at multiple maturities in the future, a Vega hedge is set up using the guarantee's 1st order derivative with respect to volatility under the Black-Scholes model.

(d) (*1.5 points*)

(i) Assess the effectiveness of the Vega hedge.

(ii) Suggest two potential ways to improve the hedge.

*The responses for all parts of this question are required on the paper provided to you.*

**12.** (*6 points*) Consider a short rate process given by

$$dr = m^{*}(r,t)dt + s(r,t)dX_t,$$

where $X_t$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.

Consider pricing functions of two traded securities with maturity at time $T$, $V(r,t)$ and $Z(r,t)$, whose risk-neutral processes are given by

$$dZ(r,t) = rZ(r,t)dt + \sigma_Z Z(r,t)dX_t,$$
$$dV(r,t) = rV(r,t)dt + \sigma_V V(r,t)dX_t,$$

where $\sigma_z = \sigma_Z(r,t)$ and $\sigma_v = \sigma_V(r,t)$ are two volatility functions.

By no arbitrage conditions, both securities satisfy Fundamental Pricing Equations,

$$rV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} m^{*}(r,t) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} s(r,t)^2$$
$$rZ = \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial r} m^{*}(r,t) + \frac{1}{2}\frac{\partial^2 Z}{\partial r^2} s(r,t)^2.$$

Consider the following renormalization using the numeraire $Z(r,t)$

$$\tilde{V}(r,t) = \frac{V(r,t)}{Z(r,t)}$$

(a) (*1.5 points*) Show, using Ito's Lemma, that

$$\frac{d\tilde{V}}{\tilde{V}} = (\sigma_Z^2 - \sigma_V \sigma_Z)dt + (\sigma_V - \sigma_Z)dX_t$$

## 12. Continued

Using Girsanov's theorem we can construct a forward risk neutral measure $\mathbb{Q}^Z$ such that $\widetilde{X}_t = X_t - \int_0^t \sigma_Z(r,u)du$ is a standard Brownian motion under $\mathbb{Q}^Z$. You are also given that $\widetilde{V}$ satisfies a partial differential equation that is similar to the Fundamental Pricing Equation, namely

$$0 = \frac{\partial \widetilde{V}}{\partial t} + \frac{\partial \widetilde{V}}{\partial r}\left(m^*(r,t) + \sigma_Z s(r,t)\right) + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial r^2} s(r,t)^2 .$$

(b) (*3 points*) Show that $\widetilde{V}$ is a martingale under $\mathbb{Q}^Z$ using:

(i) the result in part (a);

(ii) the Feynman-Kac theorem.

(c) (*1.5 points*) Derive expressions for $\sigma_Z$ and $\sigma_V$ in terms of $s(r,t)$, $V$, and $Z$.

*The responses for all parts of this question are required on the paper provided to you.*

**13.** (*6 points*) In the Vasicek model, the short rate satisfies the following stochastic differential equations (SDE):

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma dX_t$$
$$dr_t = \gamma^*(\bar{r}^* - r_t)dt + \sigma d\widetilde{X}_t$$

where $X_t$ and $\widetilde{X}_t$ are Brownian motions under the real-world measure and the risk-neutral measure, respectively.

(a) (*0.5 points*) Interpret the meanings of the three parameters $\sigma$, $\gamma$, and $\bar{r}$.

(b) (*1 point*) Recommend a method to calibrate the five parameters $\sigma$, $\gamma$, $\gamma^*$, $\bar{r}$ and $\bar{r}^*$.

You have calibrated the parameters as given below:

| $r_0$ | $\sigma$ | $\gamma$ | $\bar{r}$ | $\gamma^*$ | $\bar{r}^*$ |
|---|---|---|---|---|---|
| 0.01 | 0.05 | 0.02 | 0.03 | 0.04 | 0.05 |

Let $Z(r,t;T)$ be the time $t$ price of a zero-coupon bond that matures at time $T$, given $r_t = r$.

(c) (*1.5 points*) Calculate :

(i) the spot rate duration of a zero-coupon bond $Z(r,0;1)$;

(ii) the market value of the interest rate risk;

(iii) the risk premium associated with $Z(r,0;1)$.

**13.** **Continued**

Consider the spot rate duration for a zero-coupon bond $Z(r, 0; \tau)$.

(d) (*1 point*) Describe how the spot rate duration depends on $\bar{r}^*$ and time to maturity $\tau$.

(e) (*0.5 points*) Explain the behavior of the spot rate duration if $\bar{r}^* > 0$.

(f) (*1.5 points*) Calculate the zero-coupon bond price $Z(r_0, 0; 1)$, assuming $r_0 = 1\%$.

*The responses for all parts of this question are required on the paper provided to you.*

**14.** (*8 points*) You are evaluating a callable semi-annual coupon bond with \$100 face amount. The bond matures in 4 years from now and can be called at par 1 year from now. The stated coupon rate is 4% per annum. For simplicity, assume that a coupon was paid today. The Vasicek model with the following parameterization is selected for pricing:

$$dr_t = \gamma^*(\bar{r}^* - r_t)dt + \sigma dX_t$$

Your colleagues made the following suggestions to estimate the parameters:

Colleague 1: $\sigma$ can be estimated directly from current market data of bond yields.

Colleague 2: Estimate $\bar{r}^*$ by taking average of the short-term rates and estimate $\gamma^*$ by regressing the changes in interest rate.

(a) (*1.5 points*) Critique the suggestions and recommend alternative methods as needed.

The parameters have been estimated as below:

$\bar{r}^*$ =0.05
$\gamma^*$ =0.45
$\sigma$ =3.0%
and $r_0$ =1.0%

You use the following worksheet to calculate the callable bond price.

| $T_i$ | $A(1;T_i)$ | $B(1;T_i)$ | $K_i$ | $S_Z(1;T_i)$ | $Z(r_0,0;T_i)$ | $N(d_1(i))$ | $N(d_2(i))$ | $Call(i)$ x100 |
|---|---|---|---|---|---|---|---|---|
| 1.5 | -0.003 | 0.448 | 0.9831 | 1.09% | 0.969 | 0.6337 | 0.6296 | 0.622 |
| 2.0 | -0.010 | 0.805 | 0.9650 | 1.96% | 0.954 | 0.6370 | 0.6296 | |
| 2.5 | -0.020 | 1.091 | 0.9462 | 2.66% | 0.938 | 0.6396 | 0.6296 | 1.475 |
| 3.0 | -0.033 | 1.319 | 0.9269 | 3.21% | 0.921 | 0.6417 | 0.6296 | 1.754 |
| 3.5 | -0.049 | 1.501 | | 3.66% | | … | … | 2.070 |
| 4.0 | -0.066 | 1.646 | 0.8877 | 4.01% | 0.884 | 0.6446 | 0.6296 | 2.108 |

## 14. **Continued**

where

- $\boldsymbol{Z(r,t;T)}$ is the Vasicek bond price function at time *t*, of a zero-coupon bond paying \$1 at maturity time $T$, thus $Z(r,t;T) = \exp\left(A(t,T) - B(t,T)r_t\right)$ with $A(t,T)$ and $B(t,T)$ as given in the formula package
- $P_c(r_t,t;T)$ is the price at time t of a bond paying coupon rate *c* and \$1 at maturity date $T$
- $K_i = Z(r_K,1;T_i)$, where $r_K$ is obtained by solving $P_c(r_K,1;4) = 1$
- $S_Z(1;T_i)$ are volatilities at time 1 for zero-coupon bonds maturing at time $T_i$ with $T_i > 1$
- $d_1(i)$ and $d_2(i)$ are values defined for options expiring at time 1 on zero-coupon bonds maturing at time $T_i$ with Vasicek model as given in the formula package
- $N(x)$ is the cumulative standard normal distribution
- $Call(i)$ are call prices for options on zero-coupon bonds maturing at time $T_i$ with strike price $K_i$ and option expiry at time 1

The following values have been calculated:

$$\left(\bar{r}^* - \frac{\sigma^2}{2(\gamma^*)^2}\right) = 0.0478$$

$$\frac{\sigma^2}{4\gamma^*} = 0.0005$$

$$Z(r_0,1;1.5) = 0.993$$

(b) (*2 points*) Calculate $Call(2)$

(c) (*1.5 points*) Calculate $K_{3.5}$

(d) (*1.5 points*) Calculate the price of the embedded call option.

(e) (*1.5 points*) Calculate the current price of the callable bond.

**15.** (*6 points*) The following table shows three par-value bonds with annual coupons. The prices are derived using a bootstrapping methodology and using the same market yield curve.

| Term | Par Value | Annual Coupon | Price |
|---|---|---|---|
| 1 year | 1,000 | 10% | 1,042.65 |
| 2 years | 1,000 | 10% | 1,073.78 |
| 3 years | 1,000 | 10% | 1,081.71 |

(a) (*1 point*) Show that the one-year, two-year, and three-year zero-coupon rates are 5.5%, 6%, and 7%, respectively.

ANSWER:

Your company owns a three-year $10 million par floating rate bond that pays an annual coupon at the prevailing 12-month LIBOR.

Your company would like to hedge against declining interest rates by entering into an interest rate swap with a $10 million notional that receives fixed rate and pays a floating rate (12-month LIBOR) on an annual basis.

(b) (*0.5 points*) Calculate the fixed swap rate.

ANSWER:

You are also given:

- The 12-month LIBOR rate is 6% one year from now and 7% two years from now.
- All positive cash flows at the end of years 1, 2, and 3 from the swap are reinvested into a fund earning 3% effective annual interest.
- All negative cash flows at the end of years 1, 2, and 3 from the swap are withdrawn from the fund charging 3% effective annual interest.
- Assume that the prevailing LIBOR rate is the same as the current spot rate in (a).

(c) (*1.5 points*) Calculate the value of the fund at the end of three years.

ANSWER:

## 15. Continued

You are given the following data for bond options:

- A European call option that expires in two years and gives its holder the right to pay $0.90 for a zero-coupon bond maturing one year later (the bond matures for $1) is priced at $0.1431.

- A European put option that expires in two years and gives its holder the right to receive $0.90 for a zero-coupon bond maturing one year later (the bond matures for $1) is priced at $0.1000.

(d) (*2 points*) Determine if an arbitrage opportunity is available. If it is available, design a strategy to exploit the arbitrage opportunity.

| ANSWER: |
|---|

Your colleague suggests hedging interest rate risk with interest rate futures or forwards.

(e) (*1 point*) Describe two advantages and two disadvantages of futures over forwards.

| ANSWER: |
|---|

****END OF EXAMINATION****